 NAME: KEY

Damped Harmonic Motion

Hopefully at this point, you can derive the period of an object undergoing simple harmonic motion by applying Newton's Second Law and finding the equation of motion for the object in question. If you can't, stop reading and figure that out first, and then come back.

The derivations you have done to this point are all idealized situations with no forces trying to stop the motion. Damped harmonic motion is a more realistic model that includes a retarding force, and assumes that the retarding force is proportional to the speed of the object. Let's imagine an object of mass m is attached to a spring with constant k and constant of proportionality b for the retarding force. (Actually, this would be the model for a mass on a spring oscillating in a viscous fluid.) Newton's Second Law for this would be

$$m\ddot{x} = -kx - b\dot{x}$$

The negatives are because the spring is always pulling the object to the origin and the retarding force is always against the motion. We first rewrite the equation as

$$m\ddot{x} + b\dot{x} + kx = 0$$

It looks kind of like a quadratic equation, except we are talking about derivatives of a function in time. It turns out we will solve it by actually turning it into a quadratic; the key is remembering that the derivative of e^x is e^x. Let's make a guess that x is of the form

$$x = x_0 e^{\lambda t}$$

The two constants are the initial x position, x_0, and a time constant λ. Therefore, the derivatives of x would be

$$\dot{x} = \lambda x_0 e^{\lambda t} \text{ and } \ddot{x} = \lambda^2 x_0 e^{\lambda t}$$

Substituting these into our equation of motion (and then canceling out the x terms) we get:

$$m\lambda^2 x_0 e^{\lambda t} + b\lambda x_0 e^{\lambda t} + kx_0 e^{\lambda t} = 0$$
$$m\lambda^2 + b\lambda + k = 0$$

Now we can see why this us useful: the function x cancels out, and we are left with a real quadratic in λ. So now we can figure out what the constant λ is:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

We need the positive root - otherwise our equation of motion would not be correct as the velocity term would end up with a negative. Therefore, we can say the position as a function of time for this object would be given by the imposing looking expression:

$$x = x_0 e^{\left(\frac{-b+\sqrt{b^2-4mk}}{2m}\right)t}$$

Admittedly, this doesn't look like simple harmonic motion at all - unless you know Euler's Identity, which is $e^{i\theta} = \cos\theta + i\sin\theta$. So let's rearrange the exponent term above as follows:

$$x = x_0 e^{\left(\frac{-b}{2m} + \frac{\sqrt{b^2-4mk}}{2m}\right)t} = x_0 e^{\left(\frac{-bt}{2m}\right)} e^{\left(\sqrt{\frac{b^2}{4m^2} - \frac{4mk}{4m^2}}\right)t} = x_0 e^{\left(\frac{-bt}{2m}\right)} e^{i\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)t}$$

Now we can use Euler's Identity to rewrite the function and then take the real part of it to finally find the position as a function of time for our system:

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$$x = x_0 e^{\left(\frac{-bt}{2m}\right)} \cos\left[\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)t\right]$$

This is the position as a function of time for a damped harmonic oscillator. Notice there is an amplitude, a decaying exponential and a cosine term. The period of the cosine term would be

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

Even though the function looks messy, it is actually pretty straightforward to interpret. Think of it as an object oscillating with a constant period but the amplitude of the motion is getting smaller as time goes on. And that is what we were hoping for!

Questions:

1. What are the units for the damping constant b?

Our original equation of motion was $m\ddot{x} + b\dot{x} + kx = 0$ → so $b\dot{x}$ has to have units of "N"

So $\frac{kg \cdot m}{s^2} = [b]\frac{m}{s}$ $\boxed{[b] = kg/s}$

2. To gain some confidence that our results are correct, let's do a few things:
   a. Show that the equation is dimensionally correct. (To do that you need to show that the exponential term is ok and the cosine term is ok.)

$-\frac{bt}{2m}$ should be dimensionless, so check:

$\frac{[kg/s][s]}{[kg]} = 1$ ✓

likewise $\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\,t$ should be dimensionless

$\left(\sqrt{\frac{[N/m]}{[kg]} - \frac{[kg/s]^2}{[kg]^2}}\right)[s] = \left(\sqrt{\left[\frac{1}{s^2}\right] - \left[\frac{1}{s^2}\right]}\right)[s] = 1$ ✓

   b. What happens to the equations when there is no damping (i.e. b = 0)?

$x = x_0 \underbrace{e^{-\frac{b}{2m}t}}_{e^0 = 1} \cos\left(\sqrt{\frac{k}{m} - \underbrace{\frac{b^2}{4m^2}}_{=0}}\;t\right) \rightarrow x = x_0 \cos\left(\sqrt{\frac{k}{m}}\;t\right)$

which is just simple harmonic motion

   c. What does the damping do to the period of the motion? Does this make sense?

$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$

damping makes bottom term smaller so the period is larger. Makes sense b/c damping will make accelerations smaller, so it will take longer to cover the same distance.

   d. Does the equation give reasonable results for the extremes of time (i.e. t=0 and t very large)?

@ $t = 0$ the $e^{-\frac{b}{2m}t}$ term becomes $e^0 = 1$ → so @ $t=0$ the amplitude is just x_0

@ $t = \infty$ $e^{-\frac{b}{2m}\infty}$ becomes $e^{-\infty} \approx 0$ So the amplitude drops to 0!

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3. The equations above are often written in a slightly different format with a slightly different definition. If we define the *damping coefficient*, γ, to be $\gamma = \frac{b}{2m}$ what are the equations we just derived? (Find the position function and the equation that gives the period.)

$$x = x_0 e^{-\frac{b}{2m}t} \cos\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\, t\right) \rightarrow \boxed{x = x_0 e^{-\gamma t} \cos\left(\sqrt{\frac{k}{m} - \gamma^2}\, t\right)}$$

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} \rightarrow \boxed{T = \frac{2\pi}{\sqrt{\frac{k}{m} - \gamma^2}}}$$

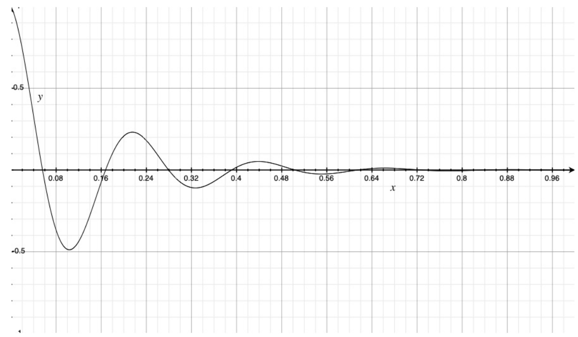
4. Use your graphing calculator to graph the position as a function of time for a 0.3 kg object attached to a spring with a constant of 250 N/m and a damping constant of $b = 4$ kg/s. Sketch it below:

$m = 0.3$ kg
$k = 250$ N/m
$b = 4$ kg/s

So $x = x_0 e^{\frac{-4}{2(.3)}t} \cos\left(\sqrt{\frac{250}{.3} - \frac{(4)^2}{4(.3)^2}}\, t\right)$

$x = x_0 e^{-6.67t} \cos(28.1t)$

call $x_0 = 1$ & graph $y = e^{-6.67x} \cos(28.1x)$



5. Try and answer the questions below by thinking about the equations, and then testing them with your graphing calculator. Think about the periods of the motion and how quickly the system decays.
   a. What happens to the motion as the mass m increases? What if it decreases instead?

   As $m\uparrow$, the period will also increase. (if $m\downarrow$ $T\downarrow$)

   As $m\uparrow$, the system will take longer to decay.

   b. What happens to the motion as the damping constant b increases? What if it decreases instead?

   As $b\uparrow$, the motion will decay faster

   As $b\uparrow$, the period will also increase. If b is too big, system won't oscillate!

   (as b goes to 0, the system will turn into simple harmonic motion)

6. The system is said to be *critically damped* when $b^2 = 4mk$. What is special about that relationship? What would be the resulting motion?

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

So $\omega = \sqrt{\frac{k}{m} - \frac{(4mk)}{4m^2}}$

$\omega = \sqrt{0}$!

This means there is NO oscillation as $\cos(0) = 1$

So the system only decays to equilibrium.

It turns out this is also the fastest way the motion decays to zero.

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7. The system is said to be *underdamped* as long as $b^2 < 4mk$. What is special about that relationship? What would be the resulting motion?

This is just the way we introduced Damped Harmonic Motion. "ω" is a + number, so it decays as it oscillates about the equilibrium position.

8. The system is said to be *overdamped* as long as $b^2 > 4mk$. What is special about that relationship? What would be the resulting motion?

if $b^2 > 4mk$, ω becomes imaginary! That means the period becomes imaginary! There is no oscillation, the system just decays to zero - but more slowly than if critically damped.

9. Find the critical damping coefficient γ in terms of m and k. What is interesting about that?

Since $\gamma = \frac{b}{2m}$ and critical damping is $b^2 = 4mk$

$$\gamma = \frac{\sqrt{4mk}}{2m}$$

$$\gamma = \sqrt{\frac{k}{m}}$$

hey! That is "ω" for the undamped oscillator (SHM)!

This frequency is often called the "natural" frequency of the oscillator.